Bellcomm/V. Mummert
MAR 26 1968
Sharon - FM files

Internal Note No. 68-FM-64

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MSC INTERNAL NOTE NO. 68-FM-64

March 8, 1968

MATCHED-CONIC SOLUTIONS TO ROUND-TRIP INTERPLANETARY TRAJECTORY PROBLEMS THAT INSURE STATE VECTOR CONTINUITY AT ALL BOUNDARIES

By Victor R. Bond,
Advanced Mission Design Branch

MISSION PLANNING AND ANALYSIS DIVISION

MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

(NASA-TM-X-69677) MATCHED-CONIC SOLUTIONS TO ROUND-TRIP INTERPLANETARY TRAJECTORY PROBLEMS THAT INSURE STATE VECTOR CONTINUITY AT ALL BOUNDARIES (NASA) 40 p

N74-70527

Unclas
00/99 16342

MSC INTERNAL NOTE NO. 68-FM-64

MATCHED-CONIC SOLUTIONS TO ROUND-TRIP INTERPLANETARY TRAJECTORY PROBLEMS THAT INSURE STATE VECTOR CONTINUITY AT ALL BOUNDARIES

By Victor R. Bond
Advanced Mission Design Branch

March 8, 1968

MISSION PLANNING AND ANALYSIS DIVISION
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
HOUSTON, TEXAS

Approved: Jack Funk
Jack Funk, Chief
Advanced Mission Design Branch

Approved: John P. Mayer
John P. Mayer, Chief
Mission Planning and Analysis Division

CONTENTS

FIGURES

MATCHED-CONIC SOLUTIONS TO ROUND-TRIP INTERPLANETARY TRAJECTORY PROBLEMS THAT INSURE STATE VECTOR CONTINUITY AT ALL BOUNDARIES

By Victor R. Bond

SUMMARY

A technique for matching conic trajectories at gravitation sphere-of-influence boundaries is presented. The match is done insuring continuity in position, velocity, and time at the sphere-of-influence boundaries. The technique is extended to several types of round-trip planetary missions and has the capability of satisfying in-flight constraints at the target planet. The types of missions considered are the free flyby, the powered flyby, and the stopover mission with a parking orbit about the target planet. An example of each of these mission types is presented.

INTRODUCTION

Matched-conic, or analytic, solutions to interplanetary trajectory problems have been used quite successfully during the past few years for mission studies (ref. 1) and as first approximation to more precise solutions. These efforts, however, have been confined mainly to solving the one-way, or single-leg, trajectory; that is, depart one planet on a certain date and arrive at the target planet a specified number of days later. In manned interplanetary applications the return trajectory must also be computed. In general, the departure and return portions of the trajectory cannot be computed independently. The two trajectories are related by boundary conditions at the target planet which cannot be specified arbitrarily.

This paper will present a technique for computing both departure and return legs of the trajectory by satisfying a set of constraints at the target planet and leaving the flight times as free variables. An important part of the technique that will be discussed is that of insuring continuity in the trajectory state at the sphere-of-influence boundaries. The solution as presented here insures continuity in position, velocity, and time at the sphere-of-influence boundaries.

By insuring continuity at all sphere-of-influence boundaries, a trajectory is obtained which may be used as a reference trajectory in, for example, navigation and guidance studies. This use distinguishes this technique from many matched-conic techniques in which planet-to-planet heliocentric conics are computed and then certain characteristics of these conics are used in meeting various mission constraints. Most techniques of this type are essentially search techniques, usually seeking out launch dates and flight times. The technique to be presented here has much of this same search capability, but has as its main purpose the production of a reference trajectory. The literature today abounds with data from programs utilizing search techniques. These data, launch dates and flight times, are used in the technique discussed in this paper as first approximations that are improved upon in the process of obtaining a more precise trajectory.

This technique has been programed for the UNIVAC 1108 digital computer in FORTRAN V. Several important types of interplanetary solutions are discussed and will be presented in the order outlined below.

The next section of this paper deals with the problem of obtaining the single-leg, matched-conic solutions between arbitrary planets. A flow chart describing the matching process for the single-leg solution is shown in figure 1.

The third section deals with the free flyby[a] solution.

The fourth section deals with the problem of obtaining a parking orbit about an oblate planet.

The fifth section explains the technique of obtaining the powered flyby[b] solution with impulse added at periapsis. This solution will also give the parking orbit solution about a spherical homogeneous planet.

All of the modes mentioned above must be solved by iteration. As a general rule these iterations occur in two phases, as shown in figure 2. The first phase, called the gross iteration phase, utilizes heliocentric conics to satisfy some constraints at the target planet. During this gross iteration phase no matching of position and velocity vectors at the spheres of influence is done. When the constraints are satisfied in the gross iteration mode, the tolerances are reduced, and the fine iteration phase begins. The fine iteration phase utilizes single leg,

---

[a]The term free flyby will refer to the situation of flying by a planet without applying a velocity change in the vicinity of the planet.

[b]The term powered flyby will refer to the situation of flying by a planet and applying a velocity change in the vicinity of the planet.

matched conics to satisfy the same set of constraints. The end result of the fine iteration is that the constraints are satisfied as well as all of the other boundary conditions.

The final section of this paper will present one example of each of the modes described above and will also give the reference from which the launch date and flight times that were used as initial guesses were taken.

SYMBOLS

a	semimajor axis
a_{ij}	elements of matrix
e	eccentricity
f	true anomaly
$\hat{h}$	unit vector along angular momentum
h	altitude above surface of planet
H	argument of hyperbolic sine or cosine (analogous to eccentric anomaly)
$\hat{i}, \hat{j}, \hat{k}$	orthogonal set of unit vectors
i	inclination with respect to planetary equator (which may be arbitrarily defined)
m	integral number
$\hat{n}$	unit vector along ascending node of orbit and planetary equator
$\underline{R}$	position vector relative to Sun
$\underline{r}$	position vector relative to planet
$\hat{S}$	unit vector along hyperbolic asymptote
T	Julian or calendar date
t	time from zero
$\underline{V}$	velocity vector relative to Sun

$\underline{v}$	velocity vector relative to planet
y	constraint function
α	right ascension
δ	declination
η	$\cos^{-1}\left(-\frac{1}{e}\right)$
μ	gravitational constant
ν	one-half angle between arrival and departure hyperbolic excess velocities at target planet
σ	$\sin^{-1} \frac{\tan \delta_\infty}{\tan i}$
ϕ	angle between vectors $\hat{n}$ and $\hat{S}$
Ω	right ascension of the ascending node

Subscripts

A	at arrival planet
D	at departure planet
k	index to resolve nodal ambiguity
P	refers to a planet
R	at return planet (usually the same as planet D, but not required)
T	at target planet
π	refers to periapsis (or minimum distance from planet)
∞	refers to hyperbolic excess vector or one of its angles

Superscripts

*	at sphere-of-influence boundary
'	evaluated from heliocentric conic
+	evaluated from planetocentric conic
(c)	computed quantity
(j)	where j = 0, 1, ..., quantity evaluated during j^{th} iteration

Special Notation

$\underline{()}$	the vector ()
$\hat{()}$	the unit vector ()
Δt	specified time increment
δt	computed time increment

DETERMINATION OF THE MATCHED-CONIC TRAJECTORY BETWEEN TWO ARBITRARY PLANETS

There are six independent variables specified in order to solve the problem of obtaining the matched-conic solution between two arbitrary planets which will be referred to as the single-leg matched conic:

T_D	the date of departure periapsis
T_A	the date of arrival periapsis
i_D	the inclination of the trajectory with respect to a planetocentric coordinate system at departure
i_A	the inclination of the trajectory with respect to a planetocentric coordinate system at arrival

$h_{\pi D}$ the periapsis altitude at departure

$h_{\pi A}$ the periapsis altitude at arrival

There are two additional inputs required to resolve the ambiguity in the right ascension of the ascending node of the trajectory with respect to each planetocentric coordinate system. These are mentioned below.

The Heliocentric Phase

If the date of arrival or departure from a planet is given, then the heliocentric position and velocity of the planet may be computed from the planetary ephemeris. At the departure planet the position and velocity vectors are $\underline{R}_{PD}(T_D)$ and $\underline{V}_{PD}(T_D)$, and at the arrival planet, $\underline{R}_{PA}(T_A)$ and $\underline{V}_{PA}(T_A)$. If the spacecraft is at a known position $\underline{r}_D(T_D)$ or $\underline{r}_A(T_A)$ from the departure or arrival planets at the same dates, then the heliocentric positions of the spacecraft are

$$\underline{R}_D(T_D) = \underline{r}_D(T_D) + \underline{R}_{PD}(T_D)$$

and (1)

$$\underline{R}_A(T_A) = \underline{r}_A(T_A) + \underline{R}_{PA}(T_A).$$

The position $\underline{R}_D$ and $\underline{R}_A$ and the time $T_A - T_D$ may be used to determine the heliocentric velocities, $\underline{V}_D$ and $\underline{V}_A$ in the vicinity of the departure and arrival planets. This problem is known as Lambert's problem and is documented in reference 2. The velocities of the spacecraft with respect to the planet are then found from

$$\underline{v}'_D(T_D) = \underline{V}_D(T_D) - \underline{V}_{PD}(T_D)$$

and (2)

$$\underline{v}'_A(T_A) = \underline{V}_A(T_A) - \underline{V}_{PA}(T_A).$$

When this procedure is used for the first time in the solution of a single-leg trajectory, $\underline{r}_D$ and $\underline{r}_A$ are zero and the spacecraft is assumed to be at the center of the planets at the dates T_D and T_A. After the planetocentric computations, good estimates for $\underline{r}^*_D$ and $\underline{r}^*_A$ are obtained. These values of $\underline{r}_D$ and $\underline{r}_A$ are superscripted (*) to indicate that they are taken at the sphere of influence of the planet.

The dates at the spheres of influence must also be corrected at arrival and departure. Therefore, the vectors

$$\underline{R}_D^*(T_D^*) = \underline{r}_D^*(T_D^*) + \underline{R}_{PD}(T_D^*)$$

and (3)

$$\underline{R}_A^*(T_A^*) = \underline{r}_A^*(T_A^*) + \underline{R}_{PA}(T_A^*),$$

where $T_D^* = T_D + t_{\pi D}$ and $T_A^* = T_A + t_{\pi A}$,

are now used in the solution of Lambert's problem to obtain $\underline{V}_D^*$ and $\underline{V}_A^*$. The velocities at the spheres of influence are

$$\underline{v}_D^{*\prime}(T_D^*) = \underline{V}_D^*(T_D^*) - \underline{V}_{PD}(T_D^*)$$

and (4)

$$\underline{v}_A^{*\prime}(T_A^*) = \underline{V}_A^*(T_A^*) - \underline{V}_{PA}(T_A^*).$$

The superscript (') indicates velocities computed from heliocentric orbits assuming massless planets.

The Planetocentric Phase

To compute the trajectory within a planetocentric sphere of influence the following is required:

$\underline{r}^*$ the position vector of the spacecraft with respect to the planet at the sphere of influence. This vector was computed during the last sphere-of-influence computation, and remained unchanged during the heliocentric phase.

$\underline{v}^{*\prime}$ the velocity vectors of the spacecraft with respect to the planet at the sphere of influence. This vector was computed during the heliocentric phase from equation (4).

i the inclination of the hyperbola with respect to the planetary equator (or some other arbitrary plane)

h_π the periapsis altitude of the hyperbola

k an index which specifies which node is to be chosen

The computation is similar for departure and arrival and the subscripts D and A are omitted except where necessary for clarity. During the first computation of the trajectory in a planetocentric phase, the position $\underline{r}^*$ is unknown except for its magnitude, and the velocity $\underline{v}^{*\prime}$ must be approximated by the velocity of the spacecraft relative to the planet and evaluated at the center of the assumed massless planet according to equation (2). This causes no special problem since $\underline{v}_D'$ and $\underline{v}_A'$ are fairly good approximations to the sphere-of-influence velocities $\underline{v}_D^{*\prime}$ and $\underline{v}_D^{*\prime}$. The only real purpose for requiring both $\underline{r}^*$ and $\underline{v}^{*\prime}$ is to allow the $\underline{v}_\infty$ vector to be computed. During the first computation in a sphere of influence the $\underline{v}_\infty$ must be approximated, while during subsequent passes, the $\underline{v}_\infty$ may be computed exactly.

a. When the spacecraft is assumed to be at the center of the planet ($\underline{r}_D$ or $\underline{r}_A = 0$), the hyperbolic asymptote is computed approximately from

$$\hat{S} = \pm\, \underline{v}'/v', \qquad (5)$$

where (+) is for departure and (-) is for arrival.

The magnitude of the hyperbolic excess velocity is computed from

$$v_\infty = \sqrt{v'^2 - 2\mu/r^*} \qquad (6)$$

b. When the spacecraft is at the sphere of influence, the hyperbolic excess velocity and asymptote may be computed exactly from

$$\underline{v}_\infty = \left\{ \frac{\underline{r}^* \cdot \underline{v}^{*\prime}}{p^* r^*} \left[1 - \cos\,(f_\infty - f^*) \right] - \frac{1}{r^*}\sqrt{\frac{\mu}{p}}\ \sin\,(f_\infty - f^*) \right\} \underline{r}^*$$

$$+ \left\{ 1 - \frac{r^*}{p^*} \left[1 - \cos\,(f_\infty - f^*) \right] \right\} \underline{v}^{*\prime} \qquad (7)$$

where

$$\cos f_\infty = \frac{-1}{e^*}\,,\quad p^* = \frac{|\underline{r}^* \times \underline{v}^{*\prime}|^2}{\mu}\,,\quad e^* = \sqrt{1 - p^*/a^*}$$

$$a^* = \left(\frac{2}{r^*} - \frac{v^{*\prime 2}}{\mu} \right)^{-1},\quad \cos f^* = \frac{1}{e^*}\left(\frac{p^*}{r^*} - 1 \right) .$$

$f_\infty - f^* > 0$ for departure

$f_\infty - f^* < 0$ for arrival

The hyperbolic excess velocity is

$$v_\infty = |\underline{v}_\infty| \tag{8}$$

and the asymptote is

$$\hat{S} = \pm\, \underline{v}_\infty / v_\infty \, . \tag{9}$$

The trajectory within the SOI is initially given of course by $\underline{r}^*$, $\underline{v}^{*\prime}$. This trajectory has orbital parameters which are in no way related to the orbital parameters which are specified. By propagating this trajectory to infinity using equation (7), the inclination and periapsis radius that would be found from $\underline{r}^*$ and $\underline{v}^{*\prime}$ lose all meaning. Therefore the specified periapsis conditions may now be imposed.

The only problem remaining in the SOI phase is to take the quantities v_∞, S, i, h_π, and r^* and determine the position and velocity vectors at periapsis $\underline{r}_\pi$ and $\underline{v}_\pi$ and the time t_π from periapsis to the SOI. These results may be easily derived from figures 3 and 4 and the results stated simply as

$$\underline{r}_{\pi D} = r_{\pi D}(\cos\eta\ \hat{S}_D - \sin\eta\ \hat{h} \times \hat{S}_D)$$

and (10)

$$\underline{v}_{\pi D} = v_{\pi D}(\sin\eta\ \hat{S}_D + \cos\eta\ \hat{h} \times \hat{S}_D)$$

for departure, and

$$\underline{r}_{\pi A} = r_{\pi A}(\cos\eta\ \hat{S}_A + \sin\eta\ \hat{h} \times \hat{S}_A)$$

and (11)

$$\underline{v}_{\pi A} = v_{\pi A}(-\sin\eta\ \hat{S}_A + \cos\eta\ \hat{h} \times \hat{S}_A)$$

for arrival where

$$r_\pi = h_\pi + \text{Radius of planet}$$

$$v_\pi = \sqrt{\mu\left(\frac{2}{r_\pi} - \frac{1}{a}\right)}$$

$$a = -\mu/v_\infty^2$$

$$\cos\eta = -\frac{1}{e}\,,\ \frac{\pi}{2} < \eta < \pi$$

$$e = 1 - {}^{r}\pi/a$$

and

$$\hat{h} = \hat{i}\,\sin\Omega\,\sin i - \hat{j}\,\cos\Omega\,\sin i + \hat{k}\,\cos i.$$

The right ascension of the ascending node Ω is ambiguous. From figures 3 and 4,

$$\Omega_k = \begin{cases} \alpha - \sigma \\ \alpha + \sigma + \pi \end{cases}$$

where α is the right ascension of the asymptote. It may also be seen that for departure the nodes which give the maximum $(k = 1)$, and minimum $(k = 0)$ periapsis declinations are

$$\Omega_1 = \alpha + \sigma + \pi$$

$$\Omega_0 = \alpha - \sigma$$

and for arrival the nodes which give the maximum and minimum periapsis declinations are

$$\Omega_1 = \alpha - \sigma$$

$$\Omega_0 = \alpha + \sigma + \pi$$

where

$$\sigma = \sin^{-1}\left(\frac{\tan\,\delta_\infty}{\tan\,i}\right)$$

$$\delta = \text{declination of the asymptote.}$$

The time from periapsis to the sphere of influence is given by

$$t_\pi = \pm\sqrt{\frac{-a^3}{\mu}}\,(e\,\sinh\,H - H) \tag{12}$$

where the plus sign is used for departure cases, and the negative sign for arrival cases. Also

$$H = \cosh^{-1}\left[\frac{1}{e}\left(1 - \frac{r^*}{a}\right)\right].$$

The position and velocity vectors at the sphere of influence satisfying the specified inclination and periapsis altitude may now be computed from

$$\underline{r}^{*+} = \left[1 - \frac{r^*(1 - \cos f)}{p}\right] \underline{r}_\pi + \frac{r^* r_\pi}{\sqrt{\mu p}} \sin f \, \underline{v}_\pi \qquad (13)$$

and

$$\underline{v}^* = \frac{-1}{r_\pi}\sqrt{\frac{\mu}{p}} \sin f \, \underline{r}_\pi + \left[1 - \frac{r_\pi}{p}(1 - \cos f)\right] \underline{v}_\pi$$

where

$$\cos f = \left(p/r^* - 1\right)\frac{1}{e} \qquad \begin{cases} \frac{\pi}{2} < f < \pi & \text{for departure} \\ \pi < f < 3\pi/2 & \text{for arrival} \end{cases}$$

$$p = a(1 - e^2)$$

Note that $\underline{r}^*$ in equation (13) has an additional superscript (+) to indicate that it differs from the original $\underline{r}^*$ of equation (7). The vector $\underline{r}^{*+}$ is used in equation (3) to determine the heliocentric position vectors of the spacecraft at the spheres of influence.

THE MATCHING PROCESS TO OBTAIN A SINGLE-LEG TRAJECTORY

The process of matching the position and velocity vectors at the spheres of influence to obtain the single-leg matched conic is essentially done by a successive approximation technique. The logic of the computational scheme is presented also in figure 1 and is given below:

a. Given the departure and arrival dates T_D and T_A, the heliocentric solution gives the relative velocity vectors $\underline{v}_D'$ and $\underline{v}_A'$, with the spacecraft assumed to be at the center of the departure and arrival planets at dates T_D and T_A, respectively.

b. Given the relative velocity vectors $\underline{v}_D'$ and $\underline{v}_A'$ or the sphere-of-influence velocities $\underline{v}_D^{*\prime}$ and $\underline{v}_A^{*\prime}$ computed in the heliocentric phase along with the inclinations i_D and i_A, the altitudes $h_{\pi D}$ and $h_{\pi A}$, the position and velocity vectors $\underline{r}_D^{*+}$, $\underline{v}_D^*$, $\underline{r}_A^{*+}$, and $\underline{v}_A^*$ at the spheres of influence are computed in the planetocentric phases. The times from each periapsis to each sphere of influence, $t_{\pi D}$ and $t_{\pi A}$, are also computed.

c. The heliocentric phase is now repeated adjusting the heliocentric positions of the spacecraft and the departure and arrival times according to equations (3). This solution gives the velocities $\underline{v}^{*\prime}_D$ and $\underline{v}^{*\prime}_A$ at the arrival and departure spheres of influence, respectively.

d. The velocity errors $|\underline{v}_D^{*\prime} - \underline{v}_D^*|$ and $|\underline{v}_A^{*\prime} - \underline{v}_A^*|$ are now computed. If they are less than the tolerance, the solution is assumed to have converged. If the tolerance is not met then steps (b) and (c) are repeated.

This process may be visualized by considering that the position vector at the sphere of influence $\underline{r}^*$ and the date T^* are changed until $\underline{v}^* = \underline{v}^{*\prime}$.

THE FREE FLYBY MATCHED CONIC MODE

The solution to this problem requires the following specifications:

T_D	departure date periapsis
i_D	inclination at the departure planet D
i_R	inclination at the return planet R

$h_{\pi D}$ periapsis altitude at D

$h_{\pi R}$ the periapsis altitude at R

$h_{\pi T}$ the periapsis altitude at the target planet T.

The flight times t_T, time from departure to target, and t_R, time from target to return, are dependent variables and require initial guesses. There is an additional constraint which must be satisfied in order to attain the free flyby, the velocity magnitudes of the arrival and departure hyperbolic excess velocities must be equal. The flight times must be computed by a numerical iteration procedure such that the flyby constraint is satisfied and the computed periapsis altitude is equal to the specified value.

The iteration is done at first using only heliocentric conics and ignoring all the boundary conditions except T_D and $h_{\pi T}$. This phase is called the gross iteration phase and is done for the purpose of improving the flight times and for determining the flyby inclination i_T and resolving the nodal ambiguity.

When the flyby constraints are approximately satisfied, the computation proceeds to the fine iteration phase. In this phase, all of the boundary conditions and the flyby constraints as well as continuity at the spheres of influence are satisfied. The trajectories used in this phase are single-leg, matched-conic solutions, whose solutions are described in a previous section. The computation for both iteration phases is illustrated schematically in figure 2.

Computation of the Free Flyby Trajectory Within the Target Planet Sphere of Influence

It is assumed that the hyperbolic excess velocities, $\underline{v}_{\infty AT}$ and $\underline{v}_{\infty DT}$, have been computed and that their magnitudes are approximately equal. These velocities may be computed by either of the methods described in the section on the planetocentric phase, depending upon whether the velocities are relative velocities and the planets are assumed to be massless or the position and velocity vectors at the sphere of influence are known.

From figure 5 it is seen that the plane of the orbit may be computed by

$$\hat{h}_T = \frac{\underline{v}_{\infty AT} \times \underline{v}_{\infty DT}}{|\underline{v}_{\infty AT} \times \underline{v}_{\infty DT}|} . \tag{14}$$

The inclination may be found from

$$\cos i_T = \hat{k} \cdot \hat{h}_T \tag{15}$$

The angle 2ν between $\underline{v}_{\infty AT}$ and $\underline{v}_{\infty DT}$ is found from

$$\sin 2\nu = \frac{|\underline{v}_{\infty AT} \times \underline{v}_{\infty DT}|}{v_{\infty AT}\, v_{\infty DT}} \tag{16}$$

The radius of periapsis is computed from

$$r_{\pi T}^{(c)} = a\ (1 - \csc\ \nu) \tag{17}$$

where

$$a = -\frac{\mu}{v_{\infty DT}^{2}} = -\frac{\mu}{v_{\infty AT}^{2}}$$

It should be emphasized that $r_{\pi T}^{(c)}$ is a computed value because in the free flyby mode $r_{\pi T}$ is actually specified. The iteration technique described in the next section will give details as to how the flight times t_T and t_R are adjusted so that the difference $r_{\pi T} - r_{\pi T}^{(c)}$ is forced to vanish.

The inclination i_T is to be used in the single-leg computations in lieu of specifying i_T. An additional quantity k is required in order to determine which of the ambiguous nodes is to be chosen. The decision as to which k to use is found by examining figures 3 and 4.

The nodal vector $\hat{n}_k$ is found from

$$\hat{n}_k = \frac{\hat{k} \times \hat{h}_T}{\sin i_T} \quad . \tag{18}$$

It is seen that for the departure leg that

$$\cos\phi_D = \hat{n}_k \cdot \hat{S}_{DT}$$

and that $\cos\phi_D$ must always be negative for a maximum periapsis declination ($k = 1$) and positive for a minimum periapsis declination ($k = 0$).

If $\cos\phi_D < 0$, $\Omega_1 = \alpha + \sigma + \pi$

If $\cos\phi_D > 0$, $\Omega_0 = \alpha - \sigma$

Similarly, for the arrival leg,

$$\cos\phi_A = \hat{n}_k \cdot \hat{S}_{AT} \ .$$

The $\cos\phi_A$ must always be positive for a maximum periapsis declination ($k = 1$) and negative for a minimum ($k = 0$).

If $\cos\phi_A > 0$, $\Omega_1 = \alpha - \sigma$.

If $\cos\phi_A < 0$, $\Omega_0 = \alpha + \sigma + \pi$

Iterative Scheme for Computing the Flight Times for the Free Flyby

In the solution for the free flyby it is required that the flight times t_T and t_R satisfy the constraints,

$$y_1 = v_{\infty AT} - v_{\infty DT} = 0 \tag{19}$$

$$y_2 = h_{\pi T} - h_{\pi T}^{(c)} = 0 \tag{20}$$

The periapsis altitude $h_{\pi T}^{(c)}$ is computed from equation (17) and $h_{\pi T}$ is specified. During the gross iteration when only the constraints in equations (19) and (20) are to be satisfied, equation (2) provides velocities which reduce the error in equation (19) to 0.1 km/hr and the error in equation (20) to 0.1 km when the iterations are completed. When this order accuracy has been obtained, the fine iteration procedure is begun and the flyby constraints and all other boundary values must now be satisfied. The fine iteration uses the more precise velocities computed from equation (6). The fine iteration is continued until the errors in equations (19) and (20) are .01 km/hr and .01 km, respectively.

The variables y_1 and y_2 may be considered functions of the flight times t_T and t_R.

$$y_1 = y_1\ (t_T,\ t_R) \tag{21}$$

$$y_2 = y_2\ (t_T,\ t_R) \tag{22}$$

The solution to these equations may be done numerically by the Newton-Raphson method as described in reference 4. Symbolically the solution may be written,

$$t_T = t_T^{(0)} + \delta t_T \tag{23}$$

and

$$t_R = t_R^{(0)} + \delta t_R \tag{24}$$

where

$$\begin{pmatrix} \delta t_T \\ \delta t_R \end{pmatrix} = \frac{-1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{pmatrix} y_1^{(0)} \\ y_2^{(0)} \end{pmatrix} \tag{25}$$

and $|A| = a_{11}a_{22} - a_{12}a_{21}$.

The matrix elements a_{ij} are defined by the partial derivatives

$$a_{11} = \frac{\partial y_1}{\partial t_T}\ ,\ a_{12} = \frac{\partial y_1}{\partial t_R}\ ,\ a_{21} = \frac{\partial y_2}{\partial t_T}\ ,\ a_{22} = \frac{\partial y_2}{\partial t_R}\ . \tag{26}$$

These derivatives are rather formidable, if not impossible, to obtain analytically, but they may be obtained numerically by the following procedure:

1. Compute a pair of trajectories (that is, one from departure planet to the target planet, and another from the target planet to the

return planet) using the first-guess times $t_T^{(0)}$ and $t_R^{(0)}$. Then form the residuals

$$y_1^{(0)} = y_1\left(t_T^{(0)}, t_R^{(0)}\right)$$

and

$$y_2^{(0)} = y_2\left(t_T^{(0)}, t_R^{(0)}\right) .$$

2. Compute a second pair of trajectories using the times $t_T^{(1)} = t_T^{(0)} + \Delta t_T$ and $t_R^{(0)}$; that is, increment $t_T^{(0)}$ by Δt_T and hold $t_R^{(0)}$ constant. Then form the residuals

$$y_1^{(1)} = y_1\left(t_T^{(1)}, t_R^{(0)}\right)$$

and

$$y_2^{(1)} = y_2\left(t_T^{(1)}, t_R^{(0)}\right).$$

3. Now compute a third pair of trajectories using the time $t_T^{(0)}$ and $t_R^{(2)} = t_R^{(0)} + \Delta t_R$ and form the residuals

$$y_1^{(2)} = y_1\left(t_T^{(0)}, t_R^{(2)}\right)$$

and

$$y_2^{(2)} = y_2\left(t_T^{(0)}, t_R^{(2)}\right) .$$

The partial derivatives may now be approximated.

$$a_{11} = \frac{y_1^{(1)} - y_1^{(0)}}{\Delta t_T} \quad , \quad a_{12} = \frac{y_1^{(2)} - y_1^{(0)}}{\Delta t_R}$$

$$a_{21} = \frac{y_2^{(1)} - y_2^{(0)}}{\Delta t_T} \quad , \quad a_{22} = \frac{y_2^{(2)} - y_2^{(0)}}{\Delta t_R}$$

4. Now compute a fourth pair of trajectories using the corrected times from equations (23) and (24). If equations (19) and (20) are still not satisfied to some tolerance, then the entire procedure may be repeated.

MATCHED-CONIC TRAJECTORY WITH A PARKING ORBIT ABOUT AN OBLATE PLANET

The solution to this problem requires the following specifications:

T_D	the departure periapsis date
T_R	the return periapsis date, or T_T, the target planet periapsis date
i_D	the inclination at the return planet R
$h_{\pi D}$	the periapsis altitude at D
$h_{\pi R}$	the periapsis altitude at R
$H_{\pi T}$	the periapsis altitude at target planet T
t_s	the stay time at the target planet

In addition, the constraint that the parking orbit be accomplished in an integral number of orbits must be imposed.

It is assumed the hyperbolic excess velocities, $\underline{v}_{\infty AT}$ and $\underline{v}_{\infty DT}$, have been computed. These velocity vectors were computed assuming that there was a specified stay time t_s at the target planet. If the radius of periapsis at the target, $r_{\pi T}$, is also specified, then it is possible to find several solutions for the parking orbit that contain $\underline{v}_{\infty AT}$ initially and $\underline{v}_{\infty DT}$ at the end of the stay time. (See fig. 6.) The rotation of the parking orbit is accomplished by the secular perturbations that arise from the target planet's oblateness. The rates of change of the node and the argument of periapsis of the parking orbit are given by

$$\dot{\Omega} = - \frac{3n\, J_2\, R_B^2}{2a^2(1 - e^2)^2} \cos i \tag{27}$$

and

$$\dot{\omega} = -\frac{3n J_2 R_B^2}{2a^2(1 - e^2)^2}\left(\frac{5}{2}\sin^2 i - 2\right) \tag{28}$$

where

$$n = \sqrt{\frac{\mu}{a^3}} .$$

None of the other elements undergo secular perturbations.

The solution is nonunique and further must be done numerically. The technique used to solve this problem is discussed at length in reference 3, and will not be repeated here. It is sufficient to say that the technique yields several pairs of inclinations and eccentricities for the parking orbit.

A unique solution to this problem may be obtained by requiring that the stay time be accomplished in an integral number of orbits,

$$t_s - mP^{(c)} = 0, \tag{29}$$

where

$$P^{(c)} = 2\pi\sqrt{\frac{r_\pi^3}{(1 - e)^3\mu}} \tag{30}$$

is the computed period.

The integer m is determined from the relation

$$m = \text{INTEGER PART of}\left(\frac{t_s}{P^{(c)}}\right) \tag{31}$$

The constraint that the stay time be accomplished in an integral number of orbits can be satisfied by permitting either t_T to be free and constraining the total flight time, or by permitting the time t_R to be free and constraining the time t_T.

Let τ denote either t_R or t_T. The constraint given by equation (29) may then be written

$$y(\tau) = t_s - mP^{(c)} = 0. \tag{32}$$

The time τ may be found iteratively by the Newton-Raphson scheme,

$$\tau^{(1)} = \tau^{(0)} - \frac{y^{(0)}(\tau)}{dy/d\tau} \quad , \tag{33}$$

where

$$\frac{dy}{d\tau} = \frac{y^{(1)} - y^{(0)}}{\Delta\tau} \quad . \tag{34}$$

This scheme requires two pairs of trajectories in order to establish the derivative, equation (34). The first iteration is a gross iteration; that is, no attempt is made to solve all of the boundary conditions or to force continuity at the spheres of influence. Succeeding iterations are fine iterations requiring all boundary conditions to be matched as well as continuity at the sphere of influence. The only boundary condition that is not specified is the inclination at the target planet which is found from equations (27) and (28).

MATCHED-CONIC TRAJECTORY WITH COINCIDENT PERIAPSIS POSITION AT THE TARGET PLANET

The solution to this problem requires the same specifications as in the last section. If the stay time is not equal to zero, then the solution yields the parking orbit about a spherical homogeneous planet, with the further restriction that the stay time, if not zero, must not be less than that of a circular orbit at the specified altitude $h_{\pi T}$.

If the stay time is specified to be zero, then the solution yields a powered flyby with impulse applied at the periapsis. Both of these modes may be conveniently handled by the same formulation since both have the common constraint that the periapsis position vectors for arrival and departure at the target are coincident.

As in the previous section for the parking orbit about an oblate planet, either t_T or t_R may be left free in order that the periapsis altitude $h_{\pi T}$ may be specified.

It is assumed that the hyperbolic excess velocity vectors $\underline{v}_{\infty AT}$ and $\underline{v}_{\infty DT}$ have been computed. The inclination and the choice of the node to be used are determined exactly as in the free flyby case. The radius of periapsis must be computed differently since the trajectory is not continuous in velocity.

The solution for the radius of periapsis starts with the requirement that the periapsides of the arrival and departure hyperbolas must be coincident. Figure 7 shows that this requirement may be expressed as

$$\eta_A + \eta_D + \kappa = 2\pi \tag{35}$$

where

$$\kappa = \cos^{-1}\left(\frac{-\underline{v}_{\infty AT}}{v_{\infty AT}} \cdot \frac{\underline{v}_{\infty DT}}{v_{\infty DT}}\right), \tag{36}$$

$$\eta_A = \cos^{-1}\left(\frac{-1}{1 + r_{\pi T}^{(c)} v_{\infty AT}^2/\mu}\right), \tag{37}$$

and

$$\eta_D = \cos^{-1}\left(\frac{-1}{1 + r_{\pi T}^{(c)} v_{\infty DT}^2/\mu}\right), \tag{38}$$

For the given flight times t_T and t_R, and stay time t_s, the computed radius of periapsis $r_{\pi T}^{(c)}$ is a function of only the velocity vectors $\underline{v}_{\infty AT}$ and $\underline{v}_{\infty DT}$. Equations (35) through (38) may be solved iteratively for $r_{\pi T}^{(c)}$ by a Newton-Raphson technique.

A second iteration is required to satisfy the constraint,

$$h_{\pi T} - h_{\pi T}^{(c)} = 0. \tag{39}$$

This constraint may be satisfied by permitting either the time t_T to be free and constraining the total time or by permitting the return time t_R to be free and constraining t_T. The computation is similar to that for the parking orbit about an oblate planet, discussed in the previous section.

SOME EXAMPLE PROBLEMS

Three examples were chosen to illustrate the use of this technique. In each example the departure date and launch times were chosen from the recent literature. The flight times were adjusted during the computation process in order to satisfy the mission constraints.

The first example, presented in table I, is an Earth-Mars-Earth free flyby case. The data for this case was chosen from reference 5. The initial guesses for flight times were found to be 130 days from Earth to Mars and 540 days for the return to Earth, and Earth departure date of 2 442 670 J.D. Table I, which presents this trajectory, indicates that these times were changed to 136.61598 days and 539.64774 days, respectively, in order to satisfy the constraints. The flyby and altitude constraints, equations (19) and (20), were satisfied to 10^{-3} fps and 10^{-4} n. mi., respectively.

The second example, presented in table II, was chosen from reference 6. This case is of the low energy, conjunction class of Mars missions. The flight times were chosen to be 330 days from Earth to Mars and 250 days for the return to Earth. The Earth departure date was 2 444 920 J.D., and the stay time at Mars was chosen to be 450 days. The oblateness of Mars was used in order to change the orientation of the parking orbit at its periapsis such that no plane changes were required. The time to return was allowed to be free in order to satisfy the integral orbit constraint, equation (32). The return time changed to 250.03270 days and the parking orbit solution chosen had an apoapsis altitude of 4671.63 n. mi.

The third example is presented in table III and was suggested by reference 7. This case is an Earth-Mars-Earth powered flyby. The time to Mars and return time to Earth were chosen as 150 days and 280 days, respectively. The total Earth trip time was constrained to be 430 days. To satisfy the periapsis altitude constraint, equation (39), of 200 n. mi., these initial flight times were changed to 215.20096 days and 214.79904 days.

CONCLUDING REMARKS

A technique for matching conic trajectories at gravitation sphere-of-influence boundaries is presented. The match is done insuring continuity in position, velocity, and time at the sphere-of-influence boundaries. The technique is extended to several types of round-trip planetary missions and has the capability of satisfying in-flight constraints at the target planet. The types of missions considered are the free flyby, the powered flyby, and the stopover mission with a parking orbit about the target planet. An example of each of these mission types is presented.

The chief advantage of the technique is its capability of being adapted to several mission types. The single-leg, matched-conic trajectory can be used to solve other type problems; for example, the optimum powered flyby. The requirement for its use in solving other problems is that the constraints be properly identified and stated mathematically. The limitations of this technique are no more severe than those for any other matched conic, that is, it representes only an approximation to the precision integrated trajectory.

TABLE I.- MARS FLYBY TRAJECTORY FOR AN EARTH DEPARTURE DATE OF SEPTEMBER 14, 1975 (2 442 670.0 JD)

Time from departure periapsis, days	Location (coordinate system)	Position vector[a,b]				Velocity vector[b], fps			
		X	Y	Z	MAG	$\dot{X}$	$\dot{Y}$	$\dot{Z}$	MAG
0.0000000	Earth periapsis (planetocentric)	2884.9752	-1531.5098	-1750.8164	3705.9361	23742.627	29756.194	13093.885	40256.593
1.7417634	Exit Earth SOI (planetocentric)	-7399.7302	417895.10	273262.11	499362.84	-580.43041	16264.251	10711.893	19483.516
1.7417634	Exit Earth SOI (heliocentric)	0.99728096	-0.12075633	0.00104555	1.0045658	10155.527	115785.17	3357.3675	116278.16
135.83074	Enter Mars SOI (heliocentric)	-0.39317506	1.5503938	0.042568700	1.6000374	-78955.564	14536.865	59.554011	80282.657
135.83074	Enter Mars SOI (planetocentric)	-303134.53	-51520.461	52285.222	311895.24	27036.637	4771.3289	-4550.4487	27828.976
136.61597	Mars periapsis (planetocentric)	160.46266	1735.9743	1066.9520	2043.9524	30934.356	1786.5980	-7559.1939	31894.635
137.40120	Exit Mars SOI (planetocentric)	296812.85	-16870.794	-94319.426	311895.24	26440.056	-1682.8089	-8517.2503	27828.977
137.40120	Exit Mars SOI (heliocentric)	-0.41441028	1.5542189	0.41691983	1.6090588	-74638.741	12878.482	-6357.9419	76008.028
675.15902	Enter Earth SOI (heliocentric)	0.47431718	-0.90451776	-0.00231529	1.0213396	83422.355	74235.051	11703.381	112281.87
675.15902	Enter Earth SOI (planetocentric)	28224.865	-349609.49	-355443.11	499362.84	-1435.6454	21949.082	22278.414	31307.364
676.26371	Earth periapsis (planetocentric)	3115.4685	1271.9415	940.00552	3493.9361	-21378.352	29017.331	31590.492	47927.013

[a]Planetocentric position vectors have units of n. mi. Heliocentric position vectors have units of A.U.

[b]Planetocentric vectors are in planetocentric equatorial system. Heliocentric vectors are in heliocentric ecliptic system.

TABLE II.- MARS STOPOVER TRAJECTORY FOR AN EARTH DEPARTURE DATE OF NOVEMBER 11, 1981 (2 444 920.0 JD)

Time from departure periapsis, days	Location (coordinate system)	Position vector[a,b]				Velocity vector[b], fps			
		X	Y	Z	MAG	$\dot{X}$	$\dot{Y}$	$\dot{Z}$	MAG
0.0000000	Earth periapsis (planetocentric)	2297.6528	2283.5117	-1793.7303	3705.9361	-29040.767	20189.566	-11440.706	37173.573
2.7475101	Exit Earth SOI (planetocentric)	-478964.17	-92465.727	106801.97	499362.84	-11306.295	-2418.3302	2684.7783	11869.653
2.7475101	Exit Earth SOI (heliocentric)	0.60602967	0.77696395	0.00166844	0.98520909	-89693.882	58951.344	3425.2568	107387.08
328.03909	Enter Mars SOI (heliocentric)	0.43817964	-1.3529420	-0.03921965	1.4226705	67791.559	31218.230	988.87569	74640.813
328.03909	Enter Mars SOI (planetocentric)	12349.581	-285618.63	124692.12	311895.24	-411.58775	9982.1511	-4491.9375	10954.006
330.00000	Mars periapsis (planetocentric)	273.65033	245.73212	-2010.5901	2043.9524	-2334.4455	18799.897	1979.9743	19047.469
330.00000	IN (Parking orbit)	273.65033	245.73212	-2010.5901	2043.9524	-1671.8119	13463.536	1417.9575	13640.834
780.00000	OUT (Parking orbit)	1690.9235	-328.63498	-1100.2355	2043.9524	7551.0782	954.25401	11320.026	13640.834
780.00000	Mars periapsis (planetocentric)	1690.9235	-328.63498	-1100.2355	2043.9524	15107.613	1909.1975	22648.231	27291.525
780.97342	Exit Mars SOI (planetocentric)	120937.98	30877.912	285830.72	311895.24	8523.5776	2243.9887	2059.3142	22405.614
780.97342	Exit Mars SOI (Heliocentric)	-1.6334542	0.29072198	0.04956455	1.6598640	-3519.6500	-63724.370	18453.965	66435.924
1028.96410	Enter Earth SOI (heliocentric)	0.96287607	-0.29456184	0.00590578	1.0069418	34963.921	98470.662	-30910.280	108969.69
1028.96410	Enter Earth SOI (planetocentric)	120592.42	-267857.40	403823.14	499362.84	8100.0748	17183.340	-26239.516	32394.298
1030.0327	Earth periapsis (planetocentric)	2956.0211	-1111.5969	-1494.6174	3493.9361	-4188.2767	34553.257	-33981.892	48643.994

[a]Planetocentric position vectors have units of n. mi. Heliocentric position vectors have units of A.U.

[b]Planetocentric vectors are in planetocentric equatorial system. Heliocentric vectors are in heliocentric ecliptic system.

TABLE III.- MARS POWERED FLYBY TRAJECTORY FOR AN EARTH DEPARTURE DATE OF MAY 18, 1971 (2 441 090.0 JD)

Time from departure periapsis, days	Location (coordinate system)	Position vector[a]				Velocity vector, fps			
		X[b]	Y	Z	MAG	$\dot{X}$	$\dot{Y}$	$\dot{Z}$	MAG
0.0000000	Earth periapsis (planetocentric)	-3658.3147	-317.39613	499.95613	3705.9361	-151.66115	-30255.901	-20317.645	36445.154
3.3876666	Exit Earth SOI (planetocentric)	430355.27	-163222.02	-193690.89	499362.84	8177.4125	-2844.3857	-3507.5349	9341.4883
3.3876666	Exit Earth SOI (heliocentric)	-0.49829549	-0.88087299	-.00139603	1.0120462	91363.338	-52984.744	-2086.6210	105636.15
212.90443	Enter Mars SOI (heliocentric)	1.1536435	0.87232474	-0.00951384	1.4463521	-40227.565	62228.977	1343.5856	74111.454
212.90443	Enter Mars SOI (planetocentric)	265911.25	158790.36	-36816.689	311895.24	-7858.1347	-4822.7766	1136.5796	9289.8475
215.20096	Mars periapsis (planetocentric)	-151.00326	-1913.9676	701.19014	2043.9524	-18035.547	761.89250	-1804.3445	18141.584
215.20096	Mars periapsis (planetocentric)	-151.00300	-1913.9675	701.19029	2043.9524	-34105.115	1440.7326	-3411.9976	34305.630
215.91641	Exit Mars SOI (planetocentric)	-305738.11	44727.758	-42453.212	311895.24	-29917.485	4590.9737	-4234.0901	30562.402
215.91641	Exit Mars SOI (heliocentric)	1.1303436	0.90208844	-0.00896249	1.4462090	-53998.810	39253.798	179.74488	66759.002
429.15961	Enter Earth SOI (heliocentric)	0.47466394	-0.89143490	.00006821	1.0099317	100538.77	7285.5982	-577.10860	100804.06
429.15961	Enter Earth SOI (planetocentric)	-195113.41	419502.75	187913.41	499362.84	15840.147	-34864.657	-15747.309	41405.703
430.00000	Earth periapsis (planetocentric)	-2575.5215	-1513.3618	-1812.1850	3493.9361	33775.970	-41335.946	-13483.446	55057.059

[a]Planetocentric position vectors have units of n. mi. Heliocentric position vectors have units of A.U.

[b]Planetocentric vectors are in planetocentric equatorial system. Heliocentric vectors are in heliocentric ecliptic system.

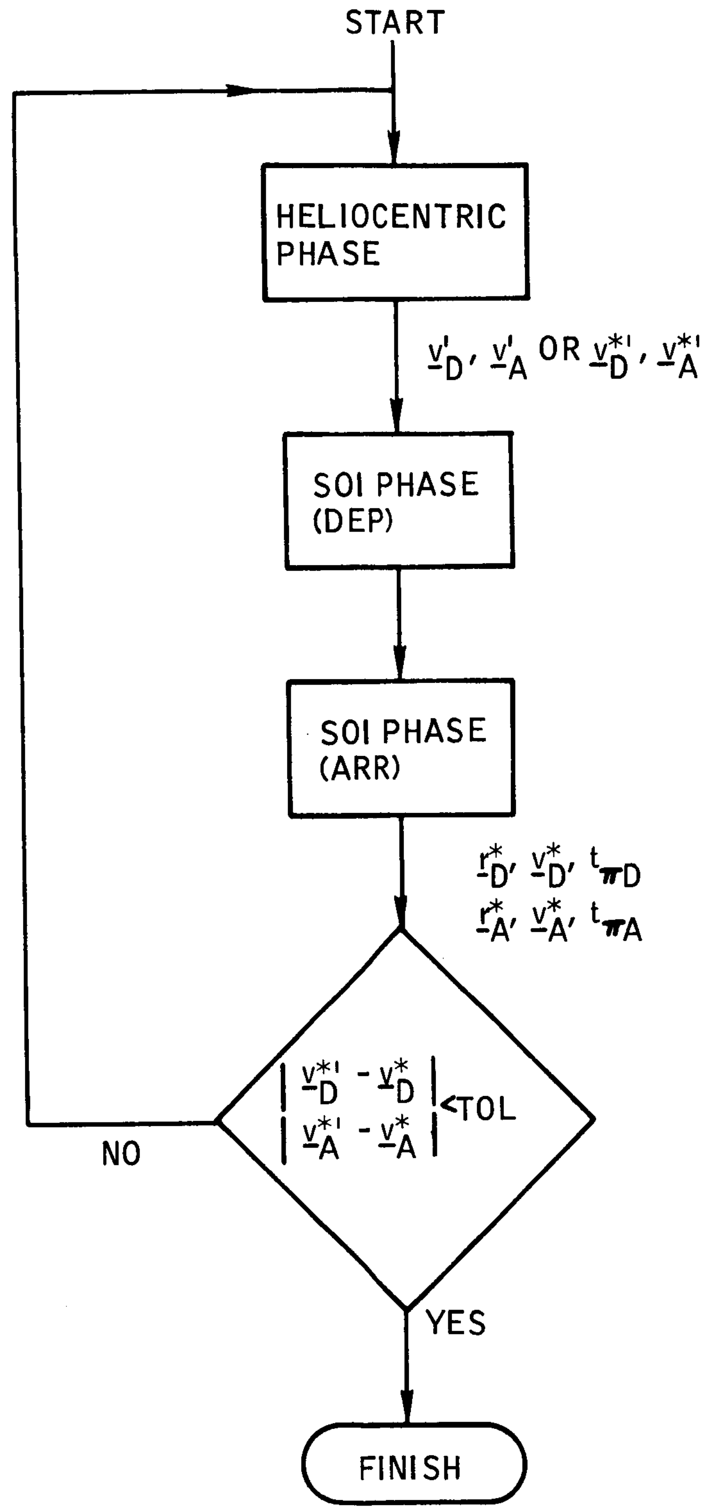


Figure 1.- Computation for a single-leg (one way), interplanetary, matched-conic trajectory.

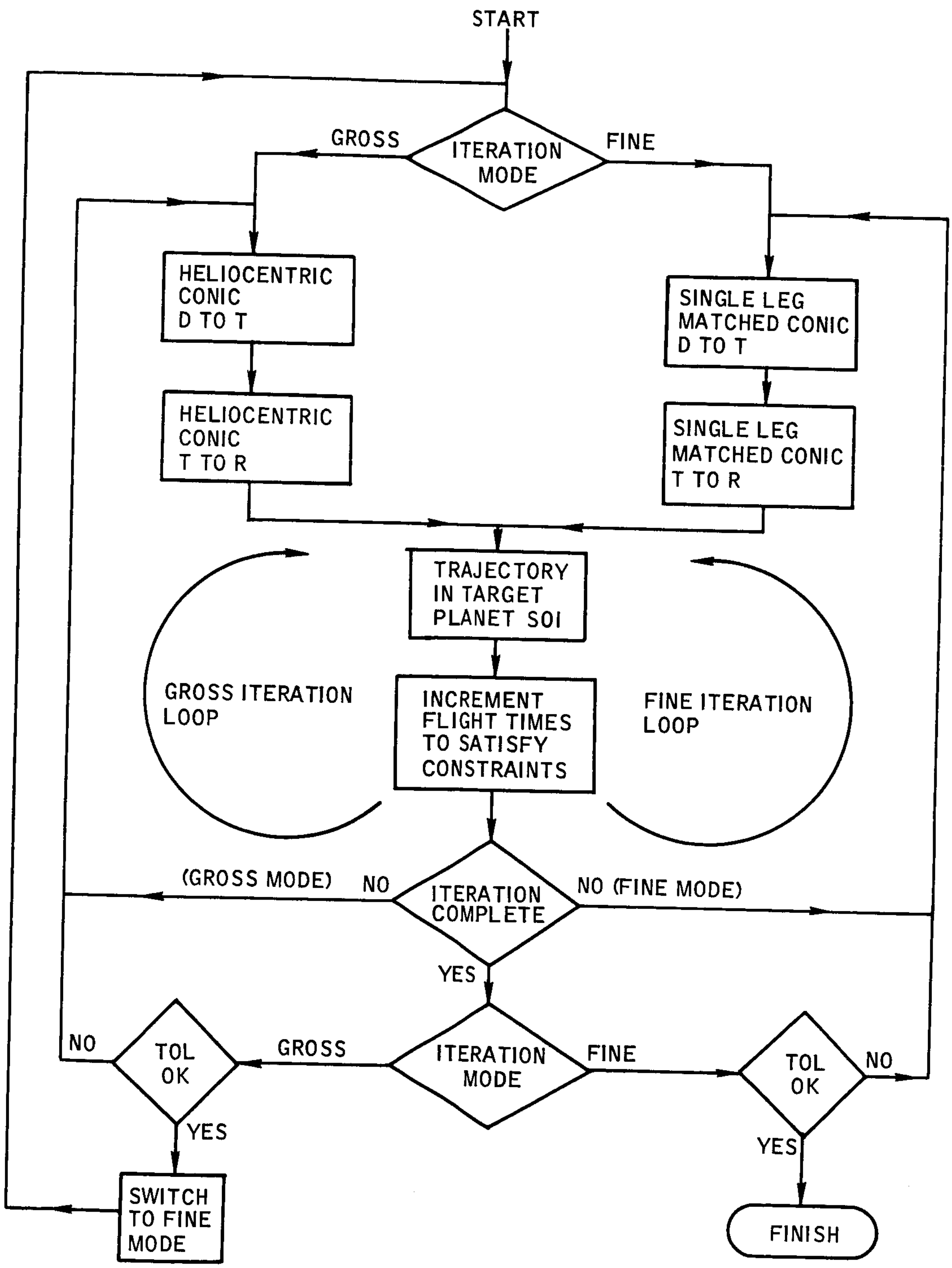
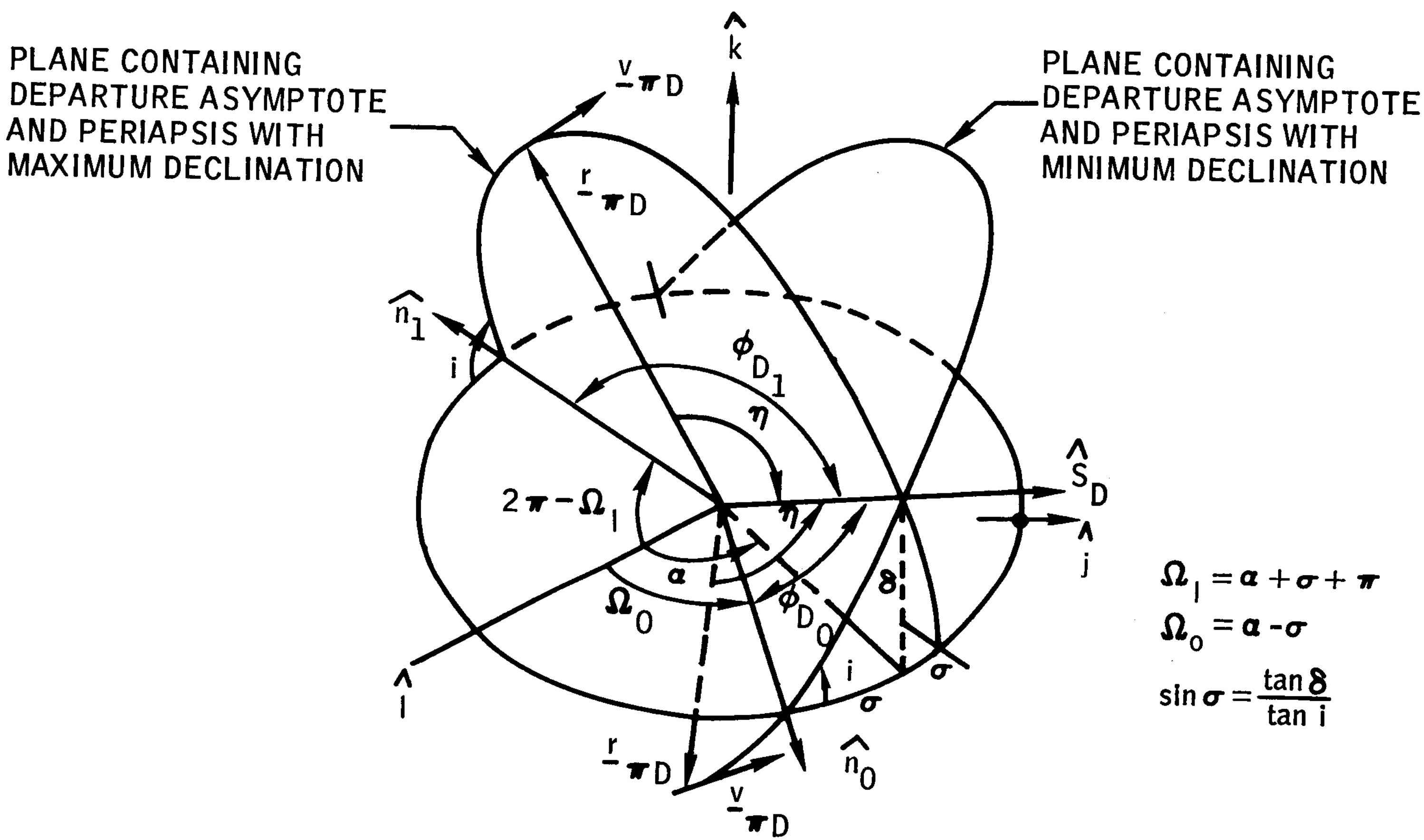


Figure 2.- Computation for round-trip interplanetary, matched-conic trajectories.



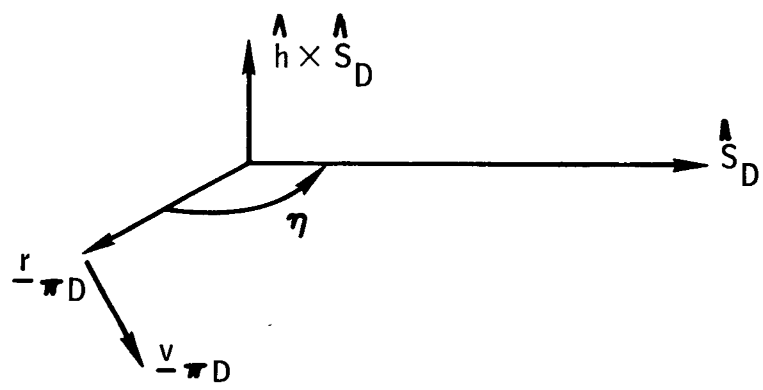
$$\Omega_I = \alpha + \sigma + \pi$$

$$\Omega_O = \alpha - \sigma$$

$$\sin\sigma = \frac{\tan\delta}{\tan i}$$

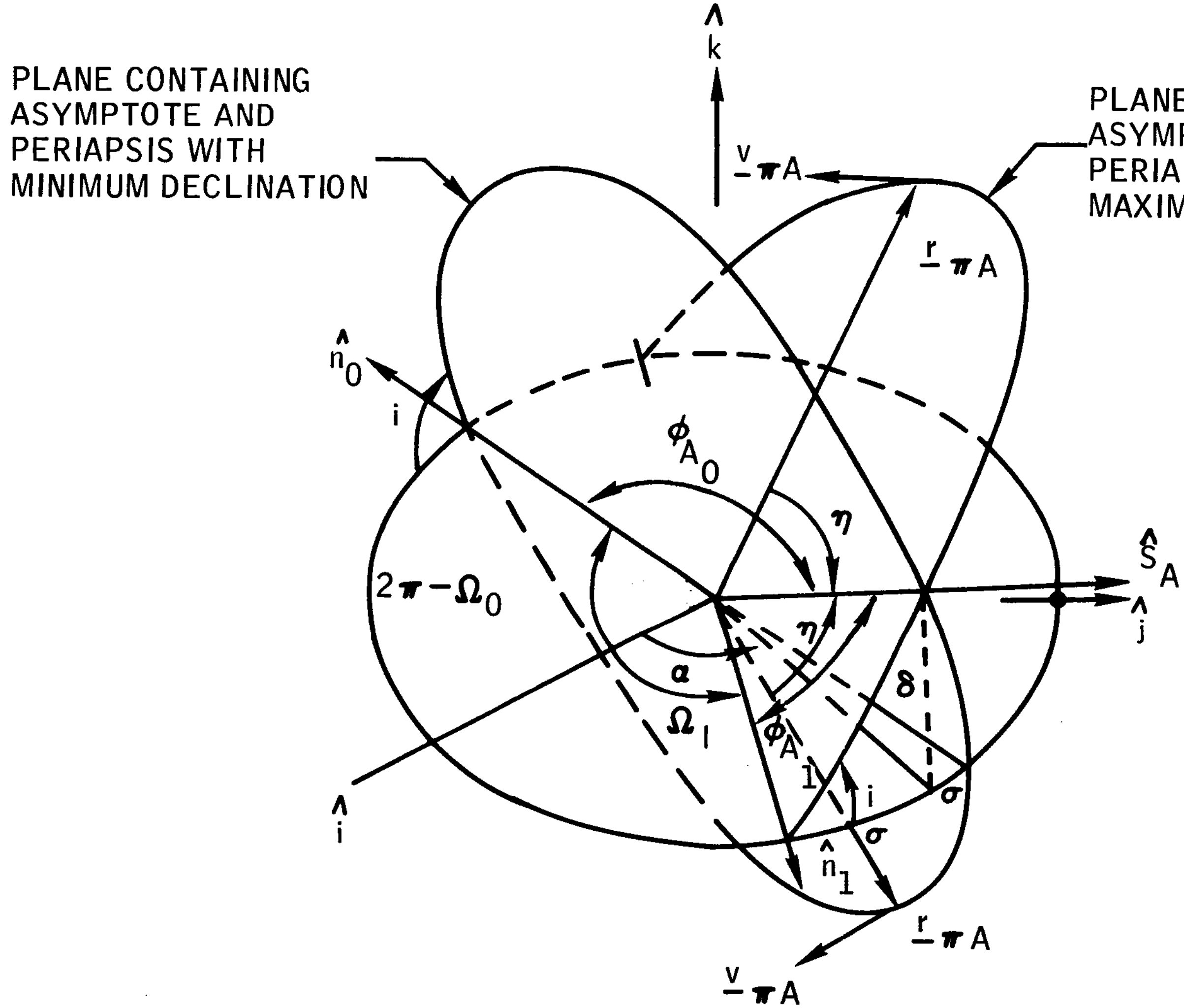
(a) The spherical geometry.

Figure 3.- Relation between the periapsis vectors and the hyperbolic asymptotes at departure.



(b) The planar geometry ($\hat{h}$ out of page).

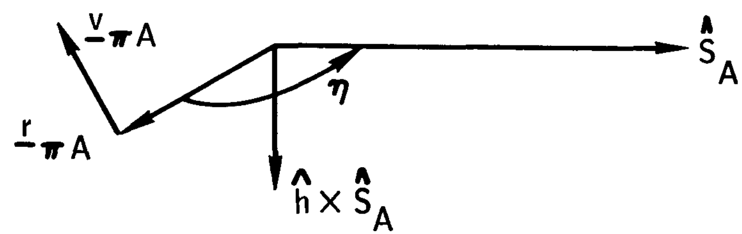
Figure 3.- Concluded.



$$\Omega_o = \alpha + \sigma + \pi$$
$$\Omega_1 = \alpha - \sigma$$
$$\sin\sigma = \frac{\tan\delta}{\tan i}$$

(a) The spherical geometry.

Figure 4.- Relation between the periapsis vector and the hyperbolic asymptotes at arrival.



(b) The planar geometry ($\hat{h}$ into page).

Figure 4.- Concluded.

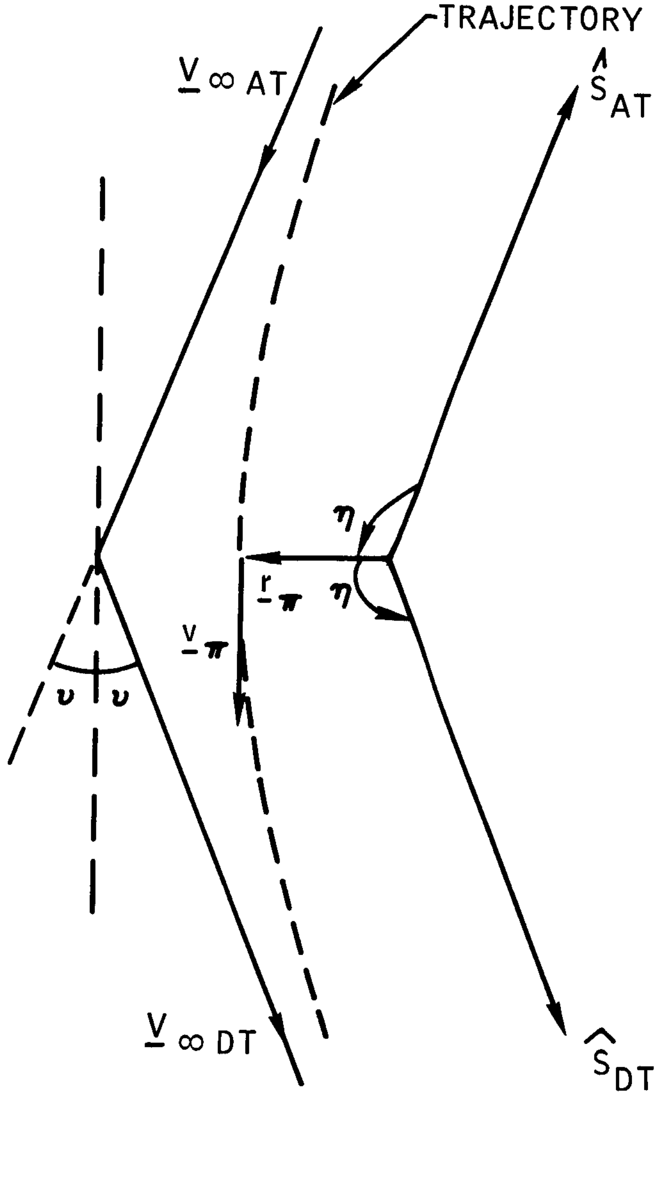


Figure 5.- Geometry of free flyby hyperbolic trajectory.

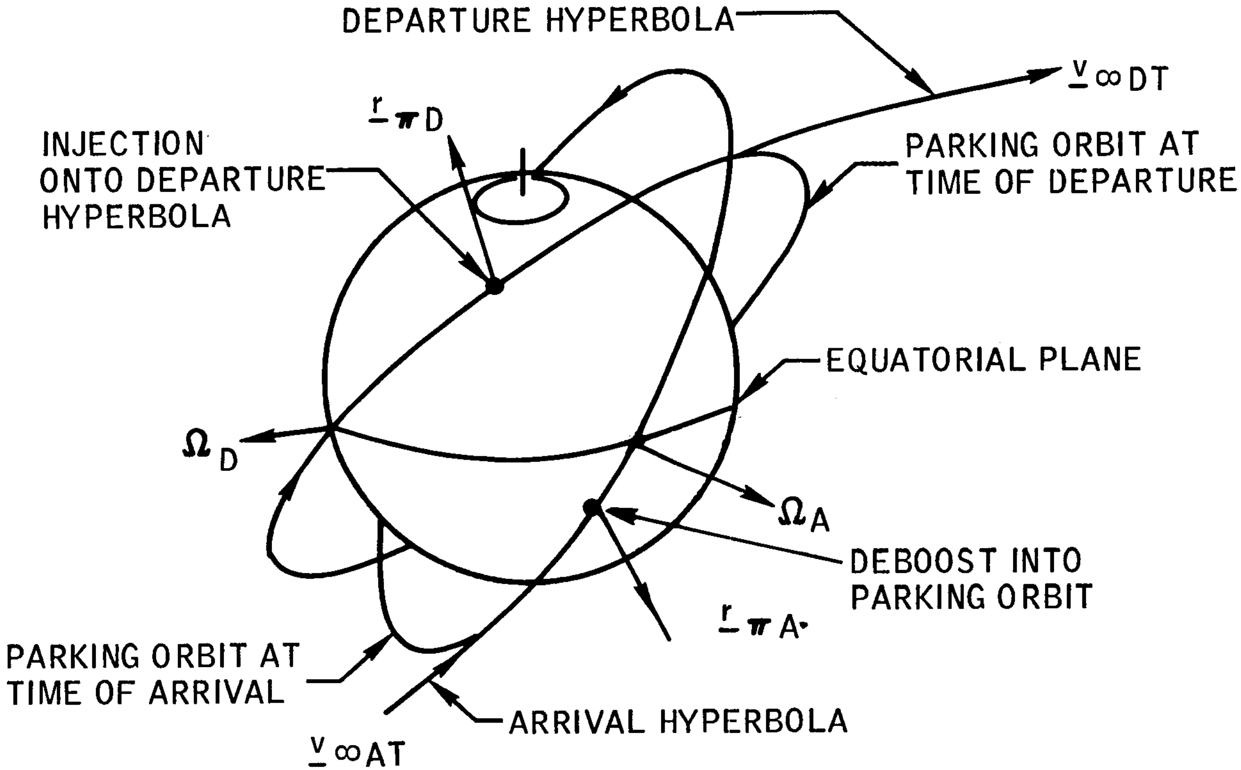


Figure 6.- Geometry of the parking orbit at the time of arrival and departure.

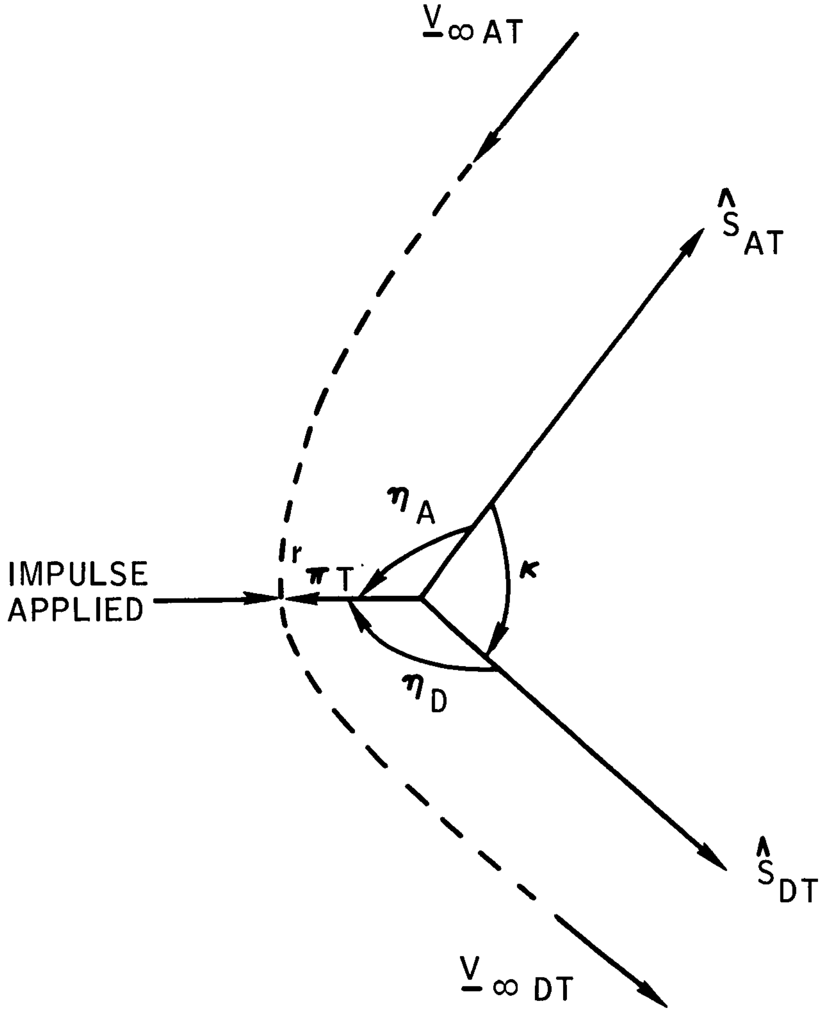


Figure 7.- Coincident periapsides condition.

REFERENCES

1. Knip, G., Jr.; and Zola, C. L.: Three-Dimensional Sphere-of-Influence Analysis of Interplanetary Trajectories to Mars. NASA TN D-1199, May 1962.

2. Battin, R. H.: Astronautical Guidance. McGraw-Hill Book Company, Inc., 1964.

3. Thibodeau, Joseph R.: Use of Planetary Oblateness for Parking Orbit Alignment. NASA TN (to be published).

4. Margenau, H.; and Murphy, G. M.: The Mathematics of Chemistry and Physics. D. VanNostrand Company, Inc., November 1955.

5. Ross, S.: Planetary Flight Handbook. NASA SP35, V3, 1963.

6. Lee, V. A.; and Wilson, S. W., Jr.: J. Spacecraft and Rockets. 4, 129-142 (1967).

7. Sohn, R. L.: Mars/Venus Flyby Missions with Manned Mars Landers. J. Spacecraft and Rockets 4, 115-117 (1967).